Introduction of Tensile Strength to Sea-Ice Modeling

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1 Introduction

The modeling of sea ice in the Arctic Ocean has been extensively studied for a number of years. These studies have proved quite useful because the variation in the ice cover of the polar oceans influences the global climate by regulating heat exchange and ocean circulation. In addition, oil companies have grown interested in finding new reserves in the Canadian Arctic and along the coast of Alaska and as a result want to study the forces exerted by moving ice floes on offshore structures. These applications have created a need for accurate and reliable models of ice flows.

The model developed by William D. Hibler in 1979 has been the standard of sea-ice modeling for several years. His model predicts the movement of packs of ice resulting from the forces of ocean currents, winds, the earth’s rotation, and the internal strength of the ice. He developed the nonlinear viscous-plastic approach, which will be subsequently explained.

This model is quite realistic in depicting sea ice in the ocean but does not account for the land-fast ice which has been observed near coastlines. We propose a change to the existing model which will bring it closer to real-world observations.

2 Logistics of the Model

2.1 Main Equations

The model is built upon three main laws which dictate that ice thickness, area and momentum must all be conserved. The first two laws are expressed as follows:

conservation of thickness: \[
\frac{\partial h}{\partial t} = -\frac{\partial uh}{\partial x} - \frac{\partial vh}{\partial y} + S_h
\]

Equation I

conservation of area: \[
\frac{\partial A}{\partial t} = -\frac{\partial uA}{\partial x} - \frac{\partial vA}{\partial y} + S_A
\]

Equation II

where: \( u \) and \( v \) are the \( x \) and \( y \)-component of ice velocity respectively
\( h \) is the average thickness of ice in a grid cell \((h \geq 0)\)
\( A \) is the fraction of area covered by sea-ice (once the area is fully covered, \( A=1 \), the ice piles up causing the height to increase additionally)
\( S_h \) and \( S_A \) are the source terms from thermodynamic parameterization (since the model discussed here does not include melting or freezing, they are 0)
Thus these two equations present the model with four unknown variables: $u$, $v$, $h$ and $A$.

The third governing equation of the model relates the effect of force on the displacement of each particular point of ice. This equation is Newton’s Second Law or the conservation of momentum equation, which stipulates that mass times acceleration in each direction must equal the sum of the forces in each direction:

$$m\frac{\partial u}{\partial t} = \sum F_x = mfv + \tau_{ax} + \tau_{wx} + F_u - mg \frac{\partial H}{\partial x}$$

and

$$m\frac{\partial v}{\partial t} = \sum F_y = -mfv + \tau_{ay} + \tau_{wy} + F_v - mg \frac{\partial H}{\partial y}$$

Equation III

where: $m$ is the mass per unit area

$f$ is the Coriolis parameter (which is necessary to account for the tendency of the ice to drift from its course because of the earth's rotation)

$\tau_{ax}$ and $\tau_{ay}$ are the x/y-component of force due to air on the top surface

$\tau_{wx}$ and $\tau_{wy}$ are the x/y-component of force due to ocean water on the bottom surface

$F_i$ is the force due to the variation in vertically integrated internal ice stress ($\sigma_{ij}$)

$g$ is the gravitational acceleration

$\partial H/\partial x$ and $\partial H/\partial y$ are the x/y-gradient of sea surface dynamic height (where $H$ is the ocean height which is calculated within the model from the ocean currents, assuming geotropic balance)

Thus the forces which affect the momentum of the ice are the Coriolis force, the force from the air, the force from the ocean, the internal ice stress and the sea surface tilt.

This analysis creates four equations (conservation of thickness, conservation of area and two conservation of momentum equations) with the four original unknown variables $u$, $v$, $h$ and $A$ and the additional unknown variables introduced in the momentum equations $\tau_{u,v}, F_u$ and $F_v$. Since four equations can only solve four unknowns the additional variables must be expressed in terms of the original unknowns and other parameters which may be set to simulate certain initial conditions of the model.
2.2 Air and Water Stress Terms

The air and ocean stress terms can be expressed in terms of constant known properties of their substances and the velocities at which they travel. Thus $\tau_a$ and $\tau_w$ can be expanded as follows:

$$\tau_{ax} = c_a \rho_a |\bar{u}_a - \bar{u}| (u_a - u)$$
$$\tau_{ax} = c_w \rho_w |\bar{u}_w - \bar{u}| (u_w - u)$$

$\tau_{ay} = c_a \rho_a |\bar{u}_a - \bar{u}| (v_a - v)$
$$\tau_{ay} = c_w \rho_w |\bar{u}_w - \bar{u}| (v_w - v)$$

where: $c_a$ and $c_w$ are the air and ocean drag coefficients
$\rho_a$ and $\rho_w$ are the air and water densities
$u_a$ and $v_a$ are the prescribed horizontal components of the air
$u_w$ and $v_w$ are the prescribed horizontal components of the ocean currents
$u$ and $v$ are the horizontal components of the ice velocity as before

and the standard Euclidian norm is $|\bar{u}_a - \bar{u}| = \sqrt{(\bar{u}_a - \bar{u})^2 + (\bar{v}_a - \bar{v})^2}$

Although often the ice velocity $(u)$ is not included in the force from air $(\tau_a)$ equation because it is generally much smaller than the air velocity $(u_a)$, the complete equations would appear as above. These expansions allow $\tau_{ax}$ and $\tau_{ay}$ to be defined in terms of observable parameters, $\bar{u}_a$ and $\bar{u}_w$, which will be prescribed by the initial experimental set-up, and the velocity of the ice $(u)$, which is iteratively recalculated in the model.

2.3 Internal Ice Stress: Ice Rheology

The relationship between the internal ice force and the rest of the variables used in the model, called ice rheology, must also be expressed in order for the model to have a solution. The force per area acting on the ice is called the stress ($\sigma$) while the ice stretching effect that results from it is called the strain ($\varepsilon$). Both the stress and the strain are composed of the component normal to the plane it acts on (called the normal stress or strain) and the component parallel to the plane (called the shear stress or strain). [See Figure 1] The first step in expanding the internal ice force can be written as follows:

$$F_i = \frac{\partial \sigma_{ij}}{\partial x} + \frac{\partial \sigma_{ij}}{\partial y}$$

$or$

$$F_u = \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{21}}{\partial y}, F_v = \frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y}$$

![Figure A](image-url)
where: $\sigma_{11}$ and $\sigma_{22}$ are the normal stresses, while $\sigma_{12}$ and $\sigma_{21}$ are the shear stresses.

The convention adopted here is such that $\sigma_{ij}$ is the stress acting on a plane which is perpendicular to the $i$-axis and in the $j$-direction, when $i$ and $j$ are not equal. In this model, since there is no internal rotation, $\sigma_{12}$ and $\sigma_{21}$ must be equivalent. This results in three unknown stress terms ($\sigma_{11}, \sigma_{22}, \sigma_{12} = \sigma_{21}$), which must be expressed in terms of the other variables.

In order to relate the stress level with its effect on the material, three different observed behaviors could be analyzed. An ideal rigid plastic material is one that does not exhibit any strain under stress until some critical point, where it separates. [See Figure 2] On the other hand, a perfectly elastic material is one that deforms non-permanently. It has a linear relationship between the amount of stress applied and the amount of strain exhibited and will then recover instantly once the stress is removed. [See Figure 3] A viscous material behaves similarly to an elastic one but the relationship is linear between the amount of stress applied and the strain rate (the change in deformation over time: $\partial \varepsilon / \partial t$). In addition, a viscous material does not recover once the stress is removed. [See Figure 4] Given these categories, it is observed that most materials behave elastically until a certain yield criteria is reached, whereupon the material behaves plastically. [See Figure 5] The yield criterion can be reached either when a single piece of ice is being compressed against itself and forced to crumble or when it is being pulled apart and divided into separate pieces. This second form of yield criterion, which is necessary to overcome in order to separate one piece of ice into two, is known as the tensile strength. Since elastic materials return quickly to the prior level of strain once a stress is removed, in order to model elastic behavior properly all prior deformations must be stored to memory. This becomes quite cumbersome for modeling purposes. The model Hibler built in 1979 avoided this difficulty by instead assuming that ice will behave in a viscous-plastic way.

2-Dimensional Charts of the Stress-Strain or Stress-Strain Rate Relationships

Plastic Behavior

- **Stress**
  - $\sigma_c$
  - **Strain Rate**

**Figure B**

Elastic Behavior

- **Stress**
  - **Strain**
  - High E = More Stiff
  - Low E = Less Stiff

**Figure C**

Viscous Behavior

- **Stress**
  - **Strain Rate**
  - High Viscosity
  - Low Viscosity

**Figure D**
The yield criteria is calculated by analyzing the stresses in different planes given by the coordinate system. Since stresses are defined in relation to the plane that passes through the point under consideration and the number of such planes is infinite because the coordinate system can be chosen arbitrarily, this would indicate that there were an infinite set of stresses at any particular point. Depending on which coordinate system is chosen, different values of $\sigma_{11}$, $\sigma_{22}$ and $\sigma_{12} = \sigma_{21}$ would be measured even though the stress state is obviously the same. Those different descriptions of the stress states lie on Mohr’s circle. [See Figure 6] Fortunately, it is always possible to choose a coordinate system where the shear stresses are zero and the normal stresses attain their maximum values, which are called either the principal stresses or the stress invariants $\sigma_1$ and $\sigma_2$. It is also convenient to depict this relationship in the absence of shear stress, i.e. graphing the yield curve as a dependence between the two normal stresses. [See Figure 7]

In the graph of the yield curve, it is apparent that the crushing yield criterion (P) beginning the plastic phase occurs at some stress value where $\sigma_1$ and $\sigma_2$ are negative (i.e. compressive) while the yield criterion necessary to separate one piece of ice into two (T) is zero. Hibler made this assumption because his model grid cells represented a relatively large-scale area where ice is too finely fragmented to resist being pulled apart.
The elliptic shape of the yield curve that Hibler chose can be defined as a function in terms of the principal stresses:

\[
F(\sigma_1, \sigma_2) = \left( \frac{\sigma_1 + \sigma_2 + P}{P} \right)^2 + \left( \frac{\sigma_2 - \sigma_1}{P} e \right)^2 - 1 = 0
\]

**Equation VI**

where: \( P \) is one endpoint of the major axis of the ellipse, which is centered at \((-P/2,-P/2)\), and the other endpoint lies at the origin,

\( e \) is the eccentricity of the ellipse, representing the relationship between the major and minor semi-axis.

This equation needs to be related to the 3-dimentional definition of \( F_u \) and \( F_v \) which are present in the momentum equation. The eigenvalues of the stress matrix must be calculated in order to express \( \sigma_i \) as a function of \( \sigma_{ij} \).

\[
\begin{vmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{vmatrix}
- \lambda I \Rightarrow \sigma_1 = \lambda_1, \sigma_2 = \lambda_2
\]

**Equation VII**

This now allows the \( F(\sigma_1, \sigma_2) \) to be transformed into \( F(\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}) \).

In order to achieve the task of expressing \( \sigma_{ij} \) as a function of the other four unknown variables presented in the conservation of thickness and conservation of area equations \((u, v, h \text{ and } A)\), the stresses must be related to the strain rate tensor. The tensor shows the deformation of the ice and is defined using the velocity \( u \) and \( v \). For a certain state of stress on the yield curve, the strain rate is defined according to the associated flow rule as follows:

\[
\dot{\epsilon}_{ij} = \gamma \frac{\partial F}{\partial \sigma_{ij}} \quad \text{where the strain rate tensor is} \quad \dot{\epsilon}_{ij} = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{\partial v}{\partial y}
\end{bmatrix}
\]

**Equation VIII**

where: \( \gamma \) is a non-negative constant

This relation is based on the idea that when the fluid flows under failure, it flows in a direction normal to the stress state on the yield curve. This allows us to find \( \dot{\epsilon}_{ij} = f(\sigma_{ij}) \), which can then be inverted to express \( \sigma_{ij} = f(\dot{\epsilon}_{ij}) \). The resulting expression for the stress is:

\[
\sigma_{ij} = \frac{P}{2\Delta} \left[ (\dot{\epsilon}_{11} + \dot{\epsilon}_{22}) + \delta_{ij} e^{-2} (\dot{\epsilon}_{11} - \dot{\epsilon}_{22}) \right] - \delta_{ij} \frac{P}{2}
\]

and

\[
\Delta = \{(\dot{\epsilon}_{11} + \dot{\epsilon}_{22})(1 + e^{-2}) + 4e^{-2}\dot{\epsilon}_{12}^2 + 2\dot{\epsilon}_{11}\dot{\epsilon}_{22}(1 - e^{-2})\}^{1/2}
\]

**Equation IX**

Based on the physics of solids, this relationship can also be expressed as:
\[ \sigma_{ij} = 2\eta \dot{e}_{ij} + \delta_{ij} (\zeta - \eta) (\dot{e}_{11} + \dot{e}_{22}) - \delta_{ij} \frac{P}{2} \]

Or
\[ \dot{e}_{ij} = \frac{1}{2\eta} \sigma_{ij} + \delta_{ij} \frac{(\eta - \zeta)}{4\eta \zeta} (\sigma_{11} + \sigma_{22}) - \delta_{ij} \frac{P}{4\zeta} \]

Equation X

Comparing the two expressions for \( \sigma_{ij} \) we find that the physical viscosities can be expressed as:
\[ \zeta = \frac{P}{2\Delta} \quad \text{and} \quad \eta = \frac{P}{2\Delta e^z} \]

Equation XI

where: \( \zeta \) is the nonlinear bulk viscosity
\( \eta \) is the nonlinear shear viscosity
\( \delta_{ij} \) is the Dirac delta function \( (\delta_{ij} = 1 \text{ when } i=j \text{ and } \delta_{ij} = 0 \text{ when } i \neq j) \)

2.4 Full Momentum Equation

Given that the stresses \( (\sigma_{ij}) \) have finally been expressed as a function of the strain tensor \( (\dot{e}_{ij}) \), which is defined by the change in ice velocities, the internal ice stress can now be expressed where the only unknown is the ice velocity. The following equations express this internal stress:

\[ F_u = \frac{\partial}{\partial x} \left[ 2\eta \frac{\partial u}{\partial x} + (\zeta - \eta) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{P}{2} \right] + \frac{\partial}{\partial y} \left[ \eta \frac{\partial u}{\partial y} + \eta \frac{\partial x}{\partial x} \right] \]

\[ F_v = \frac{\partial}{\partial y} \left[ 2\eta \frac{\partial v}{\partial y} + (\zeta - \eta) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{P}{2} \right] + \frac{\partial}{\partial x} \left[ \eta \frac{\partial v}{\partial y} + \eta \frac{\partial x}{\partial x} \right] \]

Equation XII

At this point, the momentum equation can be fully expressed in terms of the other unknowns. The air and ocean stress terms \( (\tau_{ax} \text{ and } \tau_{ay}) \) as well as the internal ice stress \( (F_i) \) have been expressed as a function of the velocities. The fully substituted momentum equations appear as follows:

\[ m \frac{\partial u}{\partial t} = m f + \tau_{ai} + \tau_{wi} + F_i - mg \frac{\partial H}{\partial x} \]

\[ m \frac{\partial u}{\partial t} = m f + \left[ c_a \rho_a \left| \vec{u}_a - \vec{u}_i \right| (u_a - u_i) \right] + \left[ c_n \rho_n \left| \vec{u}_n - \vec{u}_i \right| (u_n - u_i) \right] + \frac{\partial}{\partial x} \left[ 2\eta \frac{\partial u}{\partial x} + (\zeta - \eta) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{P}{2} \right] + \frac{\partial}{\partial y} \left[ \eta \frac{\partial u}{\partial y} + \eta \frac{\partial x}{\partial x} \right] - mg \frac{\partial H}{\partial x} \]
\[
\frac{m \partial v}{\partial t} = -mfu + \left[ c_u \rho_u \left| \ddot{u}_u - \ddot{u}_i \right| (v_u - v_i) \right] + \left[ c_w \rho_w \left| \ddot{u}_w - \ddot{u}_i \right| (v_w - v_i) \right] + \\
\frac{\partial}{\partial y} \left[ 2\gamma \frac{\partial v}{\partial y} + (\zeta - \eta) \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - P/2 \right] + \frac{\partial}{\partial x} \left[ \eta \frac{\partial u}{\partial y} + \eta \frac{\partial v}{\partial x} \right] - mg \frac{\partial H}{\partial y}
\]

Equation XIII

The combined four equations can now be used to solve for the four unknowns presented by the first two conservation equations of the model.

2.5 Elastic Addition

In order to use MATLAB (2005) to model this system, the continuous equations must be discretized. This will allow the model to iteratively compute the unknown variables every timestep. In order to capture the dynamics of ice using Hibler’s model, the timesteps between each new movement would have to be as small as thousandths of a second. This is necessary in order to allow each piece of ice to move in the direction of the force acting on it even when the movement affects the neighboring piece of ice (which may have a different direction of movement). These small timesteps make the model extremely computationally expensive.

In order to alleviate this problem, Hunke and Dukowicz (1997) suggest that an elastic component should be inserted. This would permit a small timestep (\(\Delta t_e\)) for the elastic component capturing the new velocity of the ice \((u \text{ and } v)\) and the stress \((\sigma)\) and a larger timestep (\(\Delta t_t\)) to progress every time new wind and water forces were recalculated resting in new height \((h)\) and area \((A)\) of the ice. This adds a linear stress-strain relationship for small stresses to the existing model. Hunke and Dukowicz argue that although this addition is not physically realistic, it is a useful numerical tool that greatly speeds up calculations with minimal distortions of Hibler’s theory and reasoning.

Mathematically, the addition of this elastic component comes into play in the expression of the strain rate tensor. The viscous-plastic formulation of \(\dot{\varepsilon}_{ij}\) is given as above, but with the addition of the elastic component, the equation becomes:

\[
\dot{\varepsilon}_{ij} = \frac{1}{2\eta} \sigma_{ij} + \delta_y \left( \frac{\eta - \zeta}{4\eta\zeta} \right) (\sigma_{11} + \sigma_{22}) - \delta_y \frac{P}{4\zeta} + \frac{1}{E} \frac{\partial \sigma_{ij}}{\partial t}
\]

Equation XIV

where \(E\) corresponds to Young’s modulus, which is a measure of the stiffness of a given material.
This additional component is useful because as the viscosities become infinitesimally small \((\zeta, \eta \to 0)\), in the limit the elastic component disappears leaving the original viscous-plastic definition of the stress tensor. On the other hand, as the bulk and shear viscosity approach infinity \((\zeta, \eta \to \infty)\), only the elastic component is left. The computational difficulty with the original Hibler formulation was that large viscosities required extremely small timesteps in order to avoid numerical instability. This new formulation, using elasticity, circumvents this problem.

The discrete version of the stress tensor equation with the elastic component becomes:

\[
\dot{\varepsilon}_{ij}^{k+1} = \frac{1}{2\eta} \sigma_{ij}^{k+1} + \delta_{ij} \frac{(\eta - \zeta)}{4\eta\zeta} (\sigma_{11}^{k+1} + \sigma_{22}^{k+1}) + \delta_{ij} \frac{P}{4\zeta} + \left( \frac{1}{E} \right) \frac{\sigma_{ij}^{k+1} - \sigma_{ij}^{k}}{\Delta t_e}
\]

Equation XV

where \(\Delta t_e\) is the elastic-viscous plastic (EVP) timestep equivalent to a certain fraction of the viscous-plastic (VP) timestep \(\Delta t_v\)

and the index \(k\) is the time index such that time is discretized as \(t = \Delta t_e \ast k\)

### 2.6 Full Discrete Momentum Equation

The temporally-discretized momentum equation appears as follows:

\[
\frac{m}{\Delta t_e} (u_i^{k+1} - u_i^k) = m\rho \dot{u}_i^{k+1} + \left[ \epsilon_{ij} \rho_a | \ddot{u}_j - \ddot{u}_i | (u_i - u_i^{k+1}) \right] \left[ \epsilon_{ij} \rho_w | \ddot{u}_j - \ddot{u}_i | (u_{wi} - u_{wi}^{k+1}) \right] \frac{\partial \sigma_{11}^{k+1}}{\partial x} + \frac{\partial \sigma_{12}^{k+1}}{\partial y} - mg \frac{\partial H}{\partial x}
\]

\[
\frac{m}{\Delta t_e} (v_i^{k+1} - v_i^k) = -m\rho \dot{v}_i^{k+1} + \left[ \epsilon_{ij} \rho_a | \ddot{u}_j - \ddot{u}_i | (v_i - v_i^{k+1}) \right] \left[ \epsilon_{ij} \rho_w | \ddot{u}_j - \ddot{u}_i | (v_{wi} - v_{wi}^{k+1}) \right] \frac{\partial \sigma_{21}^{k+1}}{\partial x} + \frac{\partial \sigma_{22}^{k+1}}{\partial y} - mg \frac{\partial H}{\partial y}
\]

Equation XVI

Given all of the necessary initial conditions, the solution of these equations for \(u_i^{k+1}\) and \(v_i^{k+1}\) in terms of \(u_i^k\) and \(v_i^k\) respectively supplies the model with all the necessary information to calculate the velocity of the ice at a certain time step from the previous \(u_i^k\) and \(v_i^k\).

### 3 Motivation for Change

The existing model is quite accurate in reproducing sea ice in the open ocean, however it fails to account of the land-fast ice which has been observed near coastlines. This is ice that extends out from the continent and remains in place throughout most of the winter. It is more abundant in sheltered bays and inlets or behind coastal obstructions but exists in some measure along most of the land surrounding the Arctic. Although land-fast ice only occupies a small portion of the
total area of sea ice, it is important for the local ecology as it provides a habitat for animals such as polar bears and seals.

Figure H

There are a number of potential explanations behind these observations of land-fast ice. Such reasons may include weaker currents closer to shore or shallower water lending itself to easier freezing. We suggest, by contrast, that in fact a similar strength which is necessary to overcome in order to crush ice, must also be present in order to pull the ice apart. In Hibler’s model this latter strength, called the tensile strength, is equal to zero. The addition of such a force would allow pieces of ice to bond with others, hopefully making it harder for sections of ice which have been pushed against land to rip apart.

To add tensile strength, the yield curve originally implemented by Hibler, must be moved and the equations which utilize the variables found through it must be re-derived. Essentially, this addition creates a maximum amount of stress that can be applied to the ice before it breaks apart. We have re-derived these results and added a variable tensile strength component to MATALB code that we have developed.

4 Changes: The New Internal Ice Stress Equation

The new, shifted yield curve would be:
**Yield Curve with Non-Zero Tensile Strength (Normal $\sigma_1$ vs. $\sigma_2$)**

![Diagram showing Yield Curve with Non-Zero Tensile Strength](image)

**Figure I**

The function defining this ellipse becomes:

$$F(\sigma_1, \sigma_2) = \left(\frac{\sigma_1 + \sigma_2 + P - T}{P + T}\right)^2 + \left(\frac{\sigma_2 - \sigma_1}{P + T}\right)^2 - 1 = 0$$

Equation XVII

where: $P$ and $T$ are the endpoints of the major axis of the ellipse, which is centered at some point $\left(\frac{T-P}{2}, \frac{T-P}{2}\right)$.

A new equation for the strain-rate tensor must be calculated and once again be inverted, resulting in:

$$\sigma_{ij} = \frac{P + T}{2\Delta} \left[(\dot{\epsilon}_{11} + \dot{\epsilon}_{22}) + \delta_{ij} e^{\zeta_T} (\dot{\epsilon}_{11} - \dot{\epsilon}_{22})\right] - \delta_{ij} \frac{P - T}{2}$$

Equation XVIII

Expressing this in terms of bulk and shear viscosity, it becomes:

$$\sigma_{ij} = 2\eta_T \dot{\epsilon}_{ij} + \delta_{ij} (\zeta_T - \eta_T) (\dot{\epsilon}_{11} + \dot{\epsilon}_{22}) - \delta_{ij} \frac{P - T}{2}$$

Or

$$\dot{\epsilon}_{ij} = \frac{1}{2\eta_T} \sigma_{ij} + \delta_{ij} \frac{(\eta_T - \zeta_T)}{4\eta_T \zeta_T} (\sigma_{11} + \sigma_{22}) + \delta_{ij} \frac{P - T}{4\zeta_T}$$

Equation XIX

defining $\zeta_T = \frac{P + T}{2\Delta}$ and $\eta_T = \frac{P + T}{2\Delta e^2}$

Substituting into the internal ice stress equation:

$$F_u = \frac{\partial}{\partial x} \left[2\eta_T \frac{\partial u}{\partial x} + (\zeta_T - \eta_T) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{(P - T)}{2}\right] + \frac{\partial}{\partial y} \left[\eta_T \frac{\partial u}{\partial y} + \eta_T \frac{\partial v}{\partial x}\right]$$
\[
F_v = \frac{\partial}{\partial y} \left[ 2\eta_T \frac{\partial v}{\partial y} + (\zeta_T - \eta_T)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - \frac{(P - T)}{2} \right] + \frac{\partial}{\partial x} \left[ \eta_T \frac{\partial u}{\partial y} + \eta_T \frac{\partial v}{\partial x} \right]
\]

**Equation XX**

This changes the complete momentum equation as follows:

\[
m \frac{\partial u}{\partial t} = mfv + \left[ c_u \rho \left( \bar{u}_a - \bar{u}_i \right) \right] + \left[ c_v \rho \left( \bar{u}_a - \bar{u}_j \right) \right] + \frac{\partial}{\partial x} \left[ 2\eta_T \frac{\partial u}{\partial x} + (\zeta_T - \eta_T)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - \frac{(P - T)}{2} \right] + \frac{\partial}{\partial y} \left[ \eta_T \frac{\partial u}{\partial y} + \eta_T \frac{\partial v}{\partial x} \right] - mg \frac{\partial H}{\partial x}
\]

\[
m \frac{\partial v}{\partial t} = -mfu + \left[ c_u \rho \left( \bar{u}_a - \bar{u}_i \right) \right] + \left[ c_v \rho \left( \bar{u}_a - \bar{u}_j \right) \right] + \frac{\partial}{\partial y} \left[ 2\eta_T \frac{\partial v}{\partial y} + (\zeta_T - \eta_T)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - \frac{(P - T)}{2} \right] + \frac{\partial}{\partial x} \left[ \eta_T \frac{\partial u}{\partial y} + \eta_T \frac{\partial v}{\partial x} \right] - mg \frac{\partial H}{\partial y}
\]

**Equation XXI**

In modeling our changes, we retained the elastic component proposed by Hunke and Dukowicz. In our new equation with non-zero tensile strength, the stress tensor with the elastic component appears as follows:

\[
\dot{\varepsilon}_{ij} = \frac{1}{2\eta} \sigma_{ij} + \delta_{ij} \frac{(\eta_T - \zeta_T)}{4\eta_T \zeta_T} (\sigma_{11} + \sigma_{22}) - \delta_{ij} \frac{P - T}{4\zeta_T} + \frac{1}{E} \frac{\partial \sigma_{ij}}{\partial t}
\]

**Equation XXII**

\[
\dot{\varepsilon}^k_{ij} = \frac{1}{2\eta} \sigma^k_{ij} + \delta_{ij} \frac{(\eta_T - \zeta_T)}{4\eta_T \zeta_T} (\sigma^{k+1}_{11} + \sigma^{k+1}_{22}) - \delta_{ij} \frac{P - T}{4\zeta_T} + \left( \frac{1}{E} \right) \frac{\sigma^k_{ij} - \sigma^k_{ij}}{\Delta t}
\]

**Equation XXIII**

After re-deriving all of the necessary equations for our model with the proposed changes, we implemented them in our MATLAB code to observe the impacts on ice flow.

5 Results

After implementing the above changes in the MATLAB code, we ran various experiments to see under which conditions land-fast ice may be visible. Making the tensile strength component variable, allows us to see how changing it would affect the outcome of a certain set up. We have run our model upon a 10,000 km² domain that is divided into a grid with 625 squares, each 16 km² in size [See Figure 9]. The forces of the model act on each individual grid box which in turn interacts with the surrounding boxes. The model calculates new values for the ice height (h) and area (A) at every timestep of one hour and solves the momentum equation to calculate the new ice velocity (u, v) and stress (σ) in a shorter elastic timestep every 30 seconds. As a result of the longer timestep, we plotted a view of the ice
concentration and thickness every hour. This model initialization and configuration allows us to observe the effects of specified initial wind, ocean current, land and ice conditions after a set amount of time.

For one experiment, we wanted to observe how a strip of ice would behave when wind first blows it toward a coastline and then switches direction to pull it away from the coast. [Figure 10] Initially the strip of ice occupies the central third of the grid box and is 2 meters thick. The wind originally blows southward with strength of 9 m/s and switches after four days to blow northward with the same strength. In this experiment we did not allow any ocean current to be present in order to isolate the effect of the wind. In order to simulate the conditions of a given section of a coast, we allowed the eastern and western boundaries of our model to be periodic. This allows a portion of ice which is forced out of one bound to re-enter in the opposite bound, as if the events in our view would be replicated in the sections of coast at either side. The northern boundary is open, allowing ice to drift out freely when the current and wind force it to. The southern boundary contains the shore and thus serves as an obstruction beyond which ice, wind and current cannot go. The following are snapshot pictures of the ice thickness and concentration after 7 days given that one simulation was run under the assumption that there was no tensile strength and another was run under the assumption that the tensile strength was the same as the crushing strength.
In the above figure, the sole notable difference between the tensile strength setups appears approximately 50km from the coastline where a wider thick ice concentration is present in the case where the tensile strength is equivalent to the crushing criterion. This result does not appear to support the finding of landfast ice.

We ran an additional experiment with a similar setup as above but incorporated a new ocean current criterion. The strength of this current is dependent on its distance from land. Thus the strongest current far from land is approximately 0.5 m/s while the weakest current at the shore is 0 meters per second. Since this current is present in both tensile strength setups, it allows us to observe the effects of the tensile strength rather than just the current. Similarly to the above figure, the following are snapshot pictures of the ice thickness and concentration after 7 days given that one simulation was run under the assumption that there was no tensile strength and another was run under the assumption that the tensile strength was the same as the crushing strength.
Just as the experiment with no current, these pictures do not produce land-fast ice once the wind blows the ice away from the coast. However, the ice is notably closer to the coast where non-zero tensile strength is present. In this experiment the weaker current closer to the coastline exaggerates the effects of the tensile strength. Thus, the strength creates resistance to pulling the ice away from the shore, yet not enough to create land-fast ice.

In addition to observing a particular coastline, we wanted to see how a body of water surrounded by land behaves under certain conditions. This setup could potentially simulate the Arctic Ocean which is nearly landlocked [See Figure 12]. Although the Arctic Ocean has outlets which allow water and ice to drift away, it is principally surrounded by the land masses of Eurasia, North America, Greenland, and a number of islands which allow it to be modeled as being landlocked on the large scale. Additionally discrepancies between our model and reality arise from the fact that the Arctic Ocean is 14,090,000 km² while our domain is 10,000 km². As a result of this difference, our model may not be fully accurate to replicate the effects in the Arctic Ocean, but still hopefully shows the effects of variable tensile strength in smaller bays.
Figure M

For this experiment, we initially supposed that the water was covered by ice 4/5 of a meter in thickness. We did not allow any current to be present in this body of water, but did create a wind which rotates with time. Every vector of this wind blows in the same direction, but this direction changes with time to make a full circle after 48 hours. In this experiment, each coastline acts as an obstruction similarly to the experiment above where the ice, wind and current cannot go beyond the coast. The following are pictures of ice thickens and concentration after 7 days of such conditions given that one experiment was run under the assumption that there was no tensile strength and another under the assumption that the tensile strength is the same as the crushing strength. In this particular snapshot, the time-dependent rotational wind was blowing northward.

Figure N

Note: numbers on the axes of the grid are in tens of kilometers.
By comparing these pictures, it can be observed that a higher tensile strength correlates to thicker and more concentrated regions of ice being disbursed along a higher percentage of the coastline where the tensile strength is equal to the crushing criterion than where there is no tensile strength present. The southern coastline, where the wind is ripping the ice away at the time the snapshot is taken, shows the absence of ice in both of the setups. Although real observations show that landfast ice develops along the whole coastline of the Arctic, and the pictures produced by our model to not produce such results, the case where the tensile strength is equal to the crushing criterion creates more landfast ice than where the tensile strength is zero.

In addition to the above experiments of ocean at coastlines, we also wanted to examine a setup of a single island. By creating a cross-shaped island we can observe more behavior alternatives. The concave areas of the island would allow us to observe the behavior of ice that is relatively confined while the convex areas would allow us to observe the behavior of ice with more movement freedom. In reality it has been found that ice sticks to all types of land surfaces producing landfast ice along the whole coast of a body of land.

The initial stipulations in our experiments with cross-shaped islands were similar to the case where a body of water is surrounded by land. We supposed that the whole surrounding ocean would be filled with ice 4/5 of a meter thick, did not allow any ocean current to be present, and created the same time-dependent rotational wind component. For this experiment, we allowed all of the boundaries of the domain to be periodic to assume that the surrounding ocean behaves similarly to our segment. The following is a comparison of the ice where the tensile strength is zero verses where it is equal to the crushing yield criteria ($P$). The following pictures are a view of the island where after 7 days the time-dependent rotational wind is blowing northward.

![No Tensile Strength](image1)
![T = P](image2)

Figure O

Note: numbers on the axes of the grid are in tens of kilometers
As you can see from the above pictures, land-fast ice is not present in this model along the whole coast of the island, contrary to what has been observed in reality. However, along the northern half of the coastline (where the wind is pushing the ice away from the island) there are short distances along which land-fast ice is present in the non-zero tensile strength experiment. The effect of tensile strength is even further supported by the nearly continuous presence of thick land-fast ice along the southern half of the island’s coastline in the non-zero tensile strength experiment.

6 Conclusion

The introduction of the tensile strength component into the models developed by Hibler, and by Hunke and Dukowicz was done in an effort to create land-fast ice in the outcomes of the model. Although land-fast ice is on occasion observed along all coast types in reality, our experiments do not produce such results. Despite the lack of land-fast ice along the whole body of land, the introduction of tensile strength brings the outcomes of the experiments closer to realistic observations. In the experiment with one coastline, the tensile strength served as a force that hinders the ice from being ripped apart from the land once it has been blown there. In the experiments with the cross-shaped island and the landlocked body of water, the addition of the tensile strength resulted in larger amounts of ice along a significant section of the coast. These results create a greater similarity between the model and the observed behavior of ice while maintaining the integrity of prior, accepted works.

7 References


